ON PROPERTIES OF DERIVATIVES

BY

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Introduction. Although derivatives need not be continuous, the inverse images of open intervals are heavy in another sense. Denjoy and Clarkson have shown that the inverse image of every open interval either is empty or has positive measure (see [1] and [3]). Zahorski refined this property to one of homogeneity. He proved that if x is in the inverse image of an open interval (a, b), and if $\{I_n\}$ is a sequence of closed intervals converging to x (every neighborhood of x contains all but a finite number of the closed intervals I_n) such that each of the closed intervals essentially misses the inverse image of (a, b), then the ratio of the length of I_n to the distance of I_n from x converges to 0 as n increases (see [9]). These properties together with Baire one and the Darboux property (takes connected sets to connected sets) do not classify derivatives. In fact, it is established here that these two properties are possessed by every approximate derivative and by every kth Peano derivative.

Notation and definitions. Throughout the paper, the functions considered will be real valued, measurable functions whose domains are connected subsets of the real line. The notation $E\text{-}\lim_{y\to x}$ will denote that the limit is computed only for those values of y in E. The letter μ will denote the usual Lebesgue measure on the real line.

DEFINITION 1. A function f is said to have property A if for every open interval (a, b), $f^{-1}((a, b))$ either is empty or has positive measure.

DEFINITION 2. A sequence of closed intervals $\{I_n\}$ is said to converge to x if x is not in the union of the I_n and if every neighborhood of x contains all but a finite number of the intervals I_n.

DEFINITION 3. A function f is said to have property B if for every open interval (a, b), x in $f^{-1}((a, b))$ and $\{I_n\}$ a sequence of closed intervals converging to x with

$$\mu(f^{-1}((a, b)) \cap I_n) = 0,$$

for every n, implies

$$\lim_{n\to\infty} \mu(I_n)/d(x, I_n) = 0,$$

where

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$$d(x, I_n) = \inf\{|x - y|; y \in I_n\}.$$

Observe that if a function has the Darboux property, then the inverse image of every open interval is either empty or uncountable. The first example show that if, in addition to having the Darboux property, the function is of Baire class one, it need not have property A. This example uses the concept of approximate continuity which is recalled in the next definition.

DEFINITION 4. A function f is approximately continuous if for each x in the domain of f, there is a measurable set E whose density (computed relative to the domain of f) at x is 1 such that

$$E\text{-}\lim_{y \to x} f(y) = f(x).$$

EXAMPLE 1. There is a function of Baire class one having the Darboux property but not having property A.

Proof. Let $I = [0,1]$, K be the Cantor set, $G = I - K$, and $F = \{m/2^n: m = 1, \cdots, 2^n - 1; n = 1, 2, \cdots\}$. Let $f_1: I \to I$ be the Cantor function (see [7]). Then f_1 is continuous, and $f_1(G) = F$. Let α be in $I - F$, and let $f_2: I \to I$ be an approximately continuous function such that for all x in F, $f_2(x) = 0$, and $f_2(\alpha) = 1$. The existence of such a function f_2 is proved in [9]. Let $f = f_2 \circ f_1$. Then f is a function from I onto I. It is known that an approximately continuous function is a function of Baire class one having the Darboux property (see [2]). Since the composition of a function of Baire class one with a continuous function is a function of Baire class one, f is a function of Baire class one. A continuous function has the Darboux property, and the composition of two functions having the Darboux property is a function having the Darboux property. Therefore, f has the Darboux property.

It will be shown that $f^{-1}((0,2))$ is not empty and has measure zero. By choice, α is in $I - F$. Since f_1 is continuous, there is a point β in K such that $f_1(\beta) = \alpha$. Thus $f(\beta) = f_2(\alpha) = 1$ by the choice of f_2. Hence $f^{-1}((0,2))$ is not empty. Let y be in $f^{-1}((0,2))$. Then $f(y) \neq 0$. That is, $f_2(f_1(y)) \neq 0$. So $f_1(y)$ is not in F. Since f_1 is the Cantor function, $f_1(y)$ not in F implies y is in K. Therefore $f^{-1}((0,2))$ is contained in K, and K (the Cantor set) has measure zero. Hence $f^{-1}((0,2))$ has measure zero.

It is easy to see that if f has property B, then f has property A. Example 2 shows that this implication is not reversible.

EXAMPLE 2. There is a function of Baire class one having the Darboux property and property A but not having property B.

Proof. Let $I = [0,1]$, and let $\{I_n\}$ be any sequence of pairwise disjoint, closed subintervals of I converging to 0 such that

$$\lim_{n \to \infty} \mu(I_n)/d(0, I_n) \neq 0.$$

Let h be the function on I having the following four properties:

(1) x in I_n implies $h(x) = 1$, for all n,

(2) $h(0) = 1/2$,

(3) between two consecutive closed intervals the graph of h is v-shaped. The vertex of the v is on the x-axis, and the ends are each a distance of 1 above the x-axis,

(4) $h(x) = 1$, for all other x.

Clearly, h is continuous except at 0. It follows that h is a function of Baire class one. For any x in $(0,1]$, h assumes all values in I between 0 and x. This fact together with the continuity of h on $(0,1]$ implies that h has the Darboux property. To see that h has property A, let E be the inverse image of an open interval (a,b) under h. If E is not empty, then there is an x in E such that x is in $(0,1]$. The continuity of h on $(0,1]$ then implies that E contains a nonempty, open subset of $(0,1]$, and hence E has positive measure. That h does not have property B follows since $\mu(h^{-1}((0,1)) \cap I_n) = 0$ (actually $h^{-1}((0,1)) \cap I_n$ is empty), 0 is in $h^{-1}((0,1))$, $\{I_n\}$ is a sequence of closed intervals converging to 0, but by assumption

$$\lim_{n\to\infty} \mu(I_n)/d(0,I_n) \neq 0.$$

Property A for approximate derivatives and kth Peano derivatives.

DEFINITION 5. A function f has an approximate derivative f'_{ap} if for each x in the domain of f there is a measureable set E whose density (computed relative to the domain of f) is 1 at x such that

$$E\text{-}\lim_{y\to x} (f(y) - f(x))/(y - x) = f'_{ap}(x).$$

DEFINITION 6. A function f is said to have a kth Peano derivative f_k if for each x in the domain of f there are numbers $f_1(x), \cdots, f_{k-1}(x)$ such that

$$f(x+h) = f(x) + hf_1(x) + \cdots + (h^k/k!)[f_k(x) + \epsilon(x,h)],$$

where

$$\lim_{h\to 0} \epsilon(x,h) = 0.$$

The basic properties of these two derivatives may be found in [4] and [8], respectively. The first theorem shows that both derivatives have property A.

THEOREM 1. *A function f of Baire class one has property* A *if, for every subinterval J of the domain of f on which f is bounded either above or below, f restricted to J has property* A.

Proof. Let (a,b) be an open interval, and suppose $E = f^{-1}((a,b))$ has measure zero. Let

$$E_a = \{x : f(x) \leqq a\},$$

and let

$$E_b = \{x : f(x) \geqq b\}.$$

The first step is to prove that the components of E_a and E_b are closed relative to the domain of f. Let Q be a nondegenerate component of E_a. Then Q is an interval where it is understood that the endpoints of Q may be infinite. Assume that Q has an endpoint in the domain of f; otherwise, Q is all of the domain of f. By selecting another point in Q, a subinterval $[c,d]$ of the domain of f is constructed with (c,d) contained in Q. Note that f is bounded above on $[c,d]$ by $\max[f(c),f(d),a]$. By hypothesis, f restricted to $[c,d]$ has property A. It follows that $f(c) \leqq a$ and $f(d) \leqq a$. Hence c and d are both in E_a, and therefore, Q contains the endpoint assumed to be in the domain of f. Thus Q is closed relative to the domain of f. The proof that the components of E_b are closed relative to the domain of f is similar.

Let $\{Q\}$ denote the collection of all nondegenerate components of E_a and E_b. Let P be the complement of $\bigcup \{\text{Int } Q\}$ relative to the domain of f, where Int Q denotes the interior of Q relative to the domain of f. (Interior is taken relative to the domain of f so that if x is an endpoint of the domain of f, x can not be an isolated point of P.) Then P is closed relative to the domain of f. The first step shows that no two distinct elements of $\{Q\}$ can have a common endpoint. Therefore, P is a perfect subset of the domain of f.

The next step is to establish that each x in P is in Cl E_a and in Cl E_b, where Cl E_a denotes the closure of E_a relative to the domain of f. Suppose x is not in Cl E_a. Then there is an open subinterval J of the domain of f containing x such that $E_a \cap J$ is empty. Then f is bounded below on J by a. By hypothesis, f restricted to J, which will be denoted by h, has property A. Furthermore $h^{-1}((a,b))$ is contained in E, and by assumption, E has measure zero. Hence $h^{-1}((a,b))$ has measure zero. Since h has property A, $h^{-1}((a,b))$ is empty. It follows that f is bounded below on J by b. Therefore, J is contained in E_b, and because J is open, J is contained in Int Q for some nondegenerate component Q of E_b. Since x is in J, x is in Int Q, and hence x is not in P. Thus if x is in P, x must be in Cl E_a. Similarly, if x is in P, x is in Cl E_b.

The final step is to prove that for each x in P there are two sequences $\{x_n\}$ and $\{y_n\}$ of points of P, each converging to x, such that for every n, $f(x_n) \leqq a$ and $f(y_n) \geqq b$. Let x be in P, and let n be a positive integer. Since x is in Cl E_a, there is an x_n' in $(x - 1/n,\ x + 1/n) \cap E_a$. If x_n' is in P, call it x_n. If x_n' is not in P, then x_n' is in Int Q for some nondegenerate component Q of E_a. Since x is not in Int Q, there is an endpoint x_n of Q

between x_n' and x. Because x_n is in Q, $f(x_n) \leqq a$, and since x_n is an endpoint of Q, x_n is in P. The point y_n, having the desired properties, is chosen in a similar fashion.

In view of the above statement, no x in P can be a point of continuity of f relative to P. Since f is a function of Baire class one, f has a point of continuity on every nonempty, perfect set relative to that perfect set (see [5]). Thus P is empty. By construction, E is contained in P. Hence E is empty.

COROLLARY 1.1. *Every kth Peano derivative has property* A.

Proof. In [8] it is shown that a kth Peano derivative is a function of Baire class one, and that if a kth Peano derivative is bounded either above or below on an interval, it is an ordinary kth derivative on that interval. Since ordinary derivatives have property A, kth Peano derivatives satisfy the conditions of Theorem 1.

Corollary 1.1 was first proved by H. W. Oliver in [8].

COROLLARY 1.2. *Every approximate derivative has property* A.

Proof. In [4] it is proved that an approximate derivative is a function of Baire class one, and that if an approximate derivative is bounded either above or below on an interval, it is an ordinary derivative. Since ordinary derivatives have property A, approximate derivatives satisfy the conditions of Theorem 1.

For another proof of Corollary 1.2 see [6].

Property B for approximate derivatives and kth Peano derivatives.

THEOREM 2. *Every approximate derivative has property* B.

Proof. The theorem will first be proved for a special case of property B; then it will be shown that the general case can be reduced to the special one.

Suppose f has an approximate derivative f'_{ap}. Assume that $f'_{\text{ap}}(0) > 0$, $f(0) = 0$, and $\{I_n = [a_n, b_n]\}$ is a sequence of closed intervals, with positive endpoints, converging to 0 such that for all n, x in I_n implies $f'_{\text{ap}}(x) \leqq 0$. Let E be a measurable set whose density at 0 is 1 such that

$$E\text{-}\lim_{x \to 0} f(x)/x = f'_{\text{ap}}(0).$$

The essence of the proof is to establish that

$$\lim_{n \to \infty} \mu(E \cap I_n)/b_n = 0.$$

Let $f'_{\text{ap}}(0) > \epsilon > 0$. There is a $\delta > 0$ such that $0 < |x| < \delta$ and x in E implies

$$f'_{\text{ap}}(0) - \epsilon < f(x)/x < f'_{\text{ap}}(0) + \epsilon.$$

Since $\{I_n\}$ converges to 0 there is a positive integer N such that $n > N$ implies $0 < b_n < \delta$. It suffices to show that for all $n > N$

$$\mu(E \cap I_n)/b_n < 2\epsilon/f'_{\text{ap}}(0).$$

Let $n > N$. Since f'_{ap} is bounded above by 0 on I_n, x in I_n implies $f'_{\text{ap}}(x) = f'(x)$. That is, $f'(x) \leqq 0$ for all x in I_n. Thus f is monotone nonincreasing and continuous on I_n. Let

$$A = \{x \in I_n : f'_{\text{ap}}(0) - \epsilon < f(x)/x < f'_{\text{ap}}(0) + \epsilon\}.$$

Note that $E \cap I_n$ is contained in A. If A is empty, then $E \cap I_n$ is empty, whence

$$\mu(E \cap I_n)/b_n = 0 < 2\epsilon/f'_{\text{ap}}(0).$$

If A is not empty, let

$$x_2 = \sup A.$$

Since f is continuous on I_n, $f(x)/x$ is also continuous there. Therefore,

$$f(x_2)/x_2 \geqq f'_{\text{ap}}(0) - \epsilon.$$

Let

$$x_1 = [(f'_{\text{ap}}(0) - \epsilon)/(f'_{\text{ap}}(0) + \epsilon)]x_2.$$

Since $f'_{\text{ap}}(0) > \epsilon > 0$, $0 < x_1 < x_2$. Suppose x is in I_n and $x < x_1$. Because f is monotone nonincreasing on I_n,

$$f(x) \geqq f(x_1) \geqq f(x_2).$$

Thus

$$\begin{aligned} f(x)/x &\geqq f(x_2)/x \\ &> f(x_2)/x_1 \\ &= [(f'_{\text{ap}}(0) + \epsilon)/(f'_{\text{ap}}(0) - \epsilon)]f(x_2)/x_2 \\ &\geqq f'_{\text{ap}}(0) + \epsilon. \end{aligned}$$

Hence x is not in A. It follows that A is contained in $[x_1, x_2]$. Since $E \cap I_n$ is a subset of A,

$$\begin{aligned} \mu(E \cap I_n) &\leqq x_2 - x_1 \\ &= x_2 - [(f'_{\text{ap}}(0) - \epsilon)/(f'_{\text{ap}}(0) + \epsilon)]x_2 \\ &= x_2(2\epsilon/(f'_{\text{ap}}(0) + \epsilon)) \\ &< x_2(2\epsilon/f'_{\text{ap}}(0)) \\ &\leqq b_n(2\epsilon/f'_{\text{ap}}(0)). \end{aligned}$$

As desired,

$$\mu(E \cap I_n)/b_n < 2\epsilon/f'_{ap}(0).$$

To complete the proof of the special case, it must be shown that

$$\lim_{n\to\infty} (b_n - a_n)/a_n = 0.$$

Since the density of E at 0 is 1,

$$\begin{aligned} 1 &= \lim_{n\to\infty} \mu(E \cap [0, b_n])/b_n \\ &= \lim_{n\to\infty} [\mu(E \cap [0, a_n])/b_n + \mu(E \cap I_n)/b_n] \\ &= \lim_{n\to\infty} \mu(E \cap [0, a_n])/b_n \\ &\leqq \lim_{n\to\infty} a_n/b_n \\ &\leqq 1 \end{aligned}$$

because $0 < a_n < b_n$. Therefore,

$$\lim_{n\to\infty} a_n/b_n = 1$$

and hence

$$\begin{aligned} \lim_{n\to\infty} (b_n - a_n)/a_n &= \left(\lim_{n\to\infty} b_n/a_n\right) - 1 \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

The general case is now reduced to the special one. Let (a, b) be an open interval. Suppose x is in $f'^{-1}_{ap}((a,b))$, and $\{I_n\}$ is a sequence of closed intervals converging to x with

$$\mu(f'^{-1}_{ap}((a,b)) \cap I_n) = 0,$$

for every n. By a translation of the coordinate axis system, it may be assumed that $x = 0$ and that $f(0) = 0$. Since f'_{ap} has property A, $\mu(f'^{-1}_{ap}((a,b)) \cap I_n) = 0$ implies $f'^{-1}_{ap}((a,b)) \cap I_n$ is empty. So, for each I_n, x in I_n implies either $f'_{ap}(x) \geqq b$ or $f'_{ap}(x) \leqq a$. It is shown in [4] that f'_{ap} has the Darboux property. Thus for each I_n, either $f'_{ap}(x) \geqq b$ for all x in I_n, or $f'_{ap}(x) \leqq a$ for all x in I_n. Let N denote the set of positive integers. Let

$N_1 = \{n : I_n$ has positive endpoints, and x in I_n implies $f'_{ap}(x) \geqq b\}$,

$N_2 = \{n : I_n$ has positive endpoints, and x in I_n implies $f'_{ap}(x) \leqq a\}$,

$N_3 = \{n : I_n$ has negative endpoints, and x in I_n implies $f'_{ap}(x) \geqq b\}$,

and

$$N_4 = \{n : I_n \text{ has negative endpoints, and } x \text{ in } I_n \text{ implies } f'_{ap}(x) \leqq a\}.$$

In as much as

$$N = \bigcup_{i=1}^{4} N_i,$$

to prove that

$$\lim_{n\to\infty} \mu(I_n)/d(0, I_n) = 0$$

it suffices to verify that

$$N_i\text{-}\lim_{n\to\infty} \mu(I_n)/d(0, I_n) = 0$$

whenever N_i is an infinite subset of N.

Each of these four cases easily reduces to the special situation already proved. For example, suppose N_3 is an infinite subset of N. For each n in N_3, if $I_n = [a_n, b_n]$, then let $J_n = [-b_n, -a_n]$. Since the endpoints of I_n are negative, the endpoints of J_n are positive. Let

$$g(x) = f(-x) + bx.$$

Notice that g is approximately derivable, and that

$$g'_{ap}(x) = -f'_{ap}(-x) + b,$$

from which it follows that

$$g'_{ap}(0) > 0$$

because $f'_{ap}(0) < b$. In addition, for each for the intervals J_n, x in J_n implies $-x$ is in I_n, and hence, by definition of N_3, that $f'_{ap}(-x) \leqq b$. Thus, if x is in J_n, then $g'_{ap}(x) \leqq 0$, for every n in N_3. Since the endpoints of J_n are positive, the function g and the sequence of closed intervals $\{J_n\}_{n\in N_3}$ satisfy the conditions of the special case. Therefore,

$$N_3\text{-}\lim_{n\to\infty} \mu(J_n)/d(0, J_n) = 0.$$

But by construction of J_n, $\mu(J_n) = \mu(I_n)$ and $d(0, J_n) = d(0, I_n)$. Thus

$$N_3\text{-}\lim_{n\to\infty} \mu(I_n)/d(0, I_n) = 0.$$

THEOREM 3. *Every kth Peano derivative has property* B.

Proof. As in the proof of Theorem 2, the theorem will first be proved for a special case of property B, and then the general case will be reduced to the special one.

Suppose f has a kth Peano derivative f_k. For notational purposes f is denoted by f_0. Assume that $f_0(0) = f_1(0) = \cdots = f_k(0) = 0$ and that $\{I_n = [a_n, b_n]\}$ is a sequence of closed intervals, with positive endpoints, converging to 0 such that, for each n, x in I_n implies $f_k(x) \geqq c$, where c is a fixed positive number. Let n be a positive integer. It will be shown by induction that, for each positive integer j with $1 \leqq j \leqq k$, there is a partition

$$a_n = t_{j,0} < \cdots < t_{j,m(j)} = b_n$$

of I_n such that $m(j) \leqq 2^j$ and, for each $i = 1, \cdots, m(j)$, one of the following holds for every x in $[t_{j,i-1}, t_{j,i}]$:

$1(j) : f_{k-j}(x) - f_{k-j}(t_{j,i-1}) \geqq (c/j!)(x - t_{j,i-1})^j$, and $f_{k-j}(t_{j,i-1}) \geqq 0$

$2(j) : f_{k-j}(x) - f_{k-j}(t_{j,i}) \leqq -(c/j!)(t_{j,i} - x)^j$, and $f_{k-j}(t_{j,i}) \leqq 0$

$3(j) : f_{k-j}(x) - f_{k-j}(t_{j,i-1}) \leqq -(c/j!)(x - t_{j,i-1})^j$, and $f_{k-j}(t_{j,i-1}) \leqq 0$

$4(j) : f_{k-j}(x) - f_{k-j}(t_{j,i}) \geqq (c/j!)(t_{j,i} - x)^j$, and $f_{k-j}(t_{j,i}) \geqq 0$.

To begin with, the above statement is proved for $j = 1$. Since f_k is bounded below on I_n by c, $f_k = f^{(k)}$ (the ordinary kth derivative of f) on I_n (see [8]). Thus, f'_{k-1} exists and is equal to f_k on I_n. Because $f_k(x) \geqq c > 0$ for all x in I_n, f_{k-1} is strictly monotone increasing and continuous on I_n. Thus, there is a unique point t in I_n such that

$$|f_{k-1}(t)| \leqq |f_{k-1}(x)|$$

for all x in I_n. First suppose that $t = a_n$. This assumption, together with the fact that f_{k-1} is strictly monotone increasing and continuous on I_n, implies that $f_{k-1}(a_n) \geqq 0$. By the mean value theorem for ordinary derivatives, x in I_n implies

$$f_{k-1}(x) - f_{k-1}(a_n) \geqq c(x - a_n).$$

In this case, the partition is

$$a_n = t_{1,0} < t_{1,1} = b_n$$

and, for the only subinterval involved, 1(1) holds. Next, suppose that $t = b_n$. Because f_{k-1} is strictly monotone increasing and continuous on I_n, it follows that $f_{k-1}(b_n) \leqq 0$. For every x in I_n, the mean value theorem implies that

$$f_{k-1}(x) - f_{k-1}(b_n) \leqq -c(b_n - x).$$

The partition is the same as it was for the case $t = a_n$, but this time 2(1) holds for the only subinterval concerned. Finally, assume that t is in (a_n, b_n). Necessarily, $f_{k-1}(t) = 0$. The partition is

$$a_n = t_{1,0} < t_{1,1} = t < t_{1,2} = b_n.$$

By the mean value theorem, x in $[t_{1,0}, t_{1,1}] = [a_n, t]$ yields that

$$f_{k-1}(x) - f_{k-1}(t) \leqq -c(t - x).$$

That is, 2(1) is satisfied for every x in $[t_{1,0}, t_{1,1}]$. Similarly, for every x in $[t_{1,1}, t_{1,2}] = [t, b_n]$,

$$f_{k-1}(x) - f_{k-1}(t) \geqq c(x - t),$$

and hence, 1(1) holds. Note that up to this point only possibilities 1 and

2 have occurred. It will be seen, however, that possibilities 3 and 4 arise from 2 in the induction step.

Let p be a positive integer, $1 \leqq p \leqq k-1$, and assume that the statement is true for $j = p$. Thus, there is a partition

$$a_n = t_{p,0} < \cdots < t_{p,m(p)} = b_n$$

of I_n such that $m(p) \leqq 2^p$ and, for each $i = 1, \cdots, m(p)$, either $1(p)$, $2(p)$, $3(p)$, or $4(p)$ holds for every x in $[t_{p,i-1}, t_{p,i}]$. It is enough to show that each interval $[t_{p,i-1}, t_{p,i}]$ can be divided into no more than two subintervals, on each of which either $1(p+1)$, $2(p+1)$, $3(p+1)$, or $4(p+1)$ holds for each x in it.

First let i be a positive integer, $1 \leqq i \leqq m(p)$, such that $1(p)$ holds for every x in the interval $[t_{p,i-1}, t_{p,i}]$. Since $f_{k-p}(t_{p,i-1}) \geqq 0$,

$$\begin{aligned} f_{k-p}(x) &\geqq (c/p!)(x - t_{p,i-1})^p \\ &\geqq 0. \end{aligned}$$

for all x in $[t_{p,i-1}, t_{p,i}]$. Thus, f_{k-p} is bounded below on $[t_{p,i-1}, t_{p,i}]$ by 0; wherefore, $f_{k-p} = f^{(k-p)}$ on $[t_{p,i-1}, t_{p,i}]$. Hence, $f'_{k-(p+1)}$ exists and equals f_{k-p} on $[t_{p,i-1}, t_{p,i}]$. Because

$$f_{k-p}(x) \geqq (c/p!)(x - t_{p,i-1})^p > 0$$

for all x in $(t_{p,i-1}, t_{p,i})$, $f_{k-(p+1)}$ is strictly monotone increasing and continuous on $[t_{p,i-1}, t_{p,i}]$. Therefore, there is a unique point t in $[t_{p,i-1}, t_{p,i}]$ such that

$$|f_{k-(p+1)}(t)| \leqq |f_{k-(p+1)}(x)|$$

for all x in $[t_{p,i-1}, t_{p,i}]$. If $t = t_{p,i-1}$, then the fact that $f_{k-(p+1)}$ is strictly monotone increasing and continuous implies that

$$f_{k-(p+1)}(t_{p,i-1}) \geqq 0.$$

Moreover, by the fundamental theorem of calculus, for each x in $[t_{p,i-1}, t_{p,i}]$,

$$\begin{aligned} f_{k-(p+1)}(x) - f_{k-(p+1)}(t_{p,i-1}) &= \int_{t_{p,i-1}}^{x} f_{k-p}(u)\,du \\ &\geqq (c/p!) \int_{t_{p,i-1}}^{x} (u - t_{p,i-1})^p du \\ &= (c/(p+1)!)(x - t_{p,i-1})^{p+1}. \end{aligned}$$

In this case, the interval $[t_{p,i-1}, t_{p,i}]$ is split into only one subinterval; namely, itself, and $1(p+1)$ holds for every x in $[t_{p,i-1}, t_{p,i}]$. Next, suppose that $t = t_{p,i}$. Since $f_{k-(p+1)}$ is strictly monotone increasing and continuous on $[t_{p,i-1}, t_{p,i}]$, $f_{k-(p+1)}(t_{p,i}) \leqq 0$. And by the fundamental theorem of calculus, x in $[t_{p,i-1}, t_{p,i}]$ yields that

$$\begin{aligned} f_{k-(p+1)}(t_{p,i}) - f_{k-(p+1)}(x) &= \int_x^{t_{p,i}} f_{k-p}(u)du \\ &\geqq (c/p!) \int_x^{t_{p,i}} (u - t_{p,i-1})^p du \\ &= (c/(p+1)!)[(t_{p,i} - t_{p,i-1})^{p+1} - (x - t_{p,i-1})^{p+1}] \\ &= (c/(p+1)!)[((t_{p,i} - x) + (x - t_{p,i-1}))^{p+1} - (x - t_{p,i-1})^{p+1}] \\ &\geqq (c/(p+1)!)[(t_{p,i} - x)^{p+1} + (x - t_{p,i-1})^{p+1} - (x - t_{p,i-1})^{p+1}] \\ &= (c/(p+1)!)(t_{p,i} - x)^{p+1}. \end{aligned}$$

Multiplication by -1 produces

$$f_{k-(p+1)}(x) - f_{k-(p+1)}(t_{p,i}) \leqq -(c/(p+1)!)(t_{p,i} - x)^{p+1}.$$

Again, the only subinterval into which the interval $[t_{p,i-1}, t_{p,i}]$ is divided is itself, but this time, every x in $[t_{p,i-1}, t_{p,i}]$ satisfies $2(p+1)$. Finally, assume that t is in $(t_{p,i-1}, t_{p,i})$. Consequently, $f_{k-(p+1)}(t) = 0$. In this case, the interval $[t_{p,i-1}, t_{p,i}]$ is divided into two subintervals: $[t_{p,i-1}, t]$ and $[t, t_{p,i}]$. For each x in $[t_{p,i-1}, t]$,

$$\begin{aligned} f_{k-(p+1)}(t) - f_{k-(p+1)}(x) &= \int_x^t f_{k-p}(u)du \\ &\geqq (c/p!) \int_x^t (u - t_{p,i-1})^p du \\ &= (c/(p+1)!)[(t - t_{p,i-1})^{p+1} - (x - t_{p,i-1})^{p+1}] \\ &\geqq (c/(p+1)!)(t - x)^{p+1}, \end{aligned}$$

where the last inequality is obtained by proceeding as in the above case. Multiplication by -1 results in

$$f_{k-(p+1)}(x) - f_{k-(p+1)}(t) \leqq -(c/(p+1)!)(t - x)^{p+1}.$$

Hence, $2(p+1)$ is satisfied by every x in $[t_{p,i-1}, t]$. For every x in $[t, t_{p,i}]$,

$$\begin{aligned} f_{k-(p+1)}(x) - f_{k-(p+1)}(t) &= \int_t^x f_{k-p}(u)du \\ &\geqq (c/p!) \int_t^x (u - t_{p,i-1})^p du \\ &= (c/(p+1)!)[(x - t_{p,i-1})^{p+1} - (t - t_{p,i-1})^{p+1}] \\ &\geqq (c/(p+1)!)(x - t)^{p+1}. \end{aligned}$$

In other words, $1(p+1)$ holds for every x in $[t, t_{p,i}]$.

The procedure in case $2(p)$, $3(p)$, or $4(p)$ holds is similar and the proof will be omitted.

As a result, there is a partition

$$a_n = t_0 < \cdots < t_m = b_n$$

of I_n, such that $m \leqq 2^k$, and for each $i = 1, \cdots, m$, one of the following holds for every x in $[t_{i-1}, t_i]$ (it is no longer necessary to denote f by f_0):

(1) $f(x) - f(t_{i-1}) \geqq (c/k!)(x - t_{i-1})^k$,

(2) $f(x) - f(t_i) \leqq -(c/k!)(t_i - x)^k$,

(3) $f(x) - f(t_{i-1}) \leqq -(c/k!)(x - t_{i-1})^k$,

(4) $f(x) - f(t_i) \geqq (c/k!)(t_i - x)^k$.

In each of the assertions (1), (2), (3), and (4), substitute for x the endpoint of $[t_{i-1}, t_i]$ not involved in that expression. What results, after taking absolute values, is that, for each $i = 1, \cdots, m$,

$$|f(t_i) - f(t_{i-1})| \geqq (c/k!)(t_i - t_{i-1})^k.$$

Since $f(0) = f_1(0) = \cdots = f_k(0) = 0$,

$$f(x) = (x^k/k!)\epsilon(x),$$

where

$$\lim_{x \to 0} \epsilon(x) = 0.$$

Therefore,

$$(c/k!)(t_i - t_{i-1})^k \leqq |(t_i^k/k!)\epsilon(t_i) - (t_{i-1}^k/k!)\epsilon(t_{i-1})|.$$

Multiplying this statement by $(k!/c)$ and dividing it by t_i^k produces

$$\begin{aligned}[(t_i - t_{i-1})/t_i]^k &\leqq (1/c)\,|\,\epsilon(t_i) - (t_{i-1}/t_i)^k \epsilon(t_{i-1})\,| \\ &\leqq (2/c) \sup_{x \in I_n} \epsilon(x).\end{aligned}$$

This estimate permits the following approximation:

$$\begin{aligned}(b_n - a_n)/b_n &= \sum_{i=1}^{m} (t_i - t_{i-1})/b_n \\ &< \sum_{i=1}^{m} (t_i - t_{i-1})/t_i \\ &\leqq m(2/c)^{1/k} \left(\sup_{x \in I_n} \epsilon(x) \right)^{1/k} \\ &\leqq 2^k (2/c)^{1/k} \left(\sup_{x \in I_n} \epsilon(x) \right)^{1/k}.\end{aligned}$$

Because I_n converges to 0 and because $\lim_{x \to 0} \epsilon(x) = 0$,

$$\lim_{n \to \infty} \sup_{x \in I_n} \epsilon(x) = 0.$$

Consequently,

$$\lim_{n\to\infty} (b_n - a_n)/b_n = 0.$$

But, $(b_n - a_n)/b_n = 1 - (a_n/b_n)$. Therefore,

$$\lim_{n\to\infty} a_n/b_n = 1,$$

and hence

$$\lim_{n\to\infty} (b_n - a_n)/a_n = \left(\lim_{n\to\infty} b_n/a_n\right) - 1$$
$$= 1 - 1 = 0.$$

This completes the proof of the special case.

The procedure for reducing the general case to the special one is the same as in the proof of Theorem 2. Let x be a point in $f_k^{-1}((a,b))$, and let $\{I_n\}$ be a sequence of closed intervals converging to x such that

$$\mu(f_k^{-1}((a,b)) \cap I_n) = 0,$$

for every interval I_n. By a translation, it may be assumed that $x=0$. Since f_k has property B, $\mu(f_k^{-1}((a,b)) \cap I_n) = 0$ implies

$$f_k^{-1}((a,b)) \cap I_n = \emptyset,$$

for every I_n. Thus, x in I_n implies either $f_k(x) \geqq b$ or $f_k(x) \leqq a$. Because f_k has the Darboux property, for each I_n, either $f_k(x) \geqq b$ for all x in I_n, or $f_k(x) \leqq a$ for all x in I_n. Let N denote the set of positive integers. Let

$$N_1 = \{n : I_n \text{ has positive endpoints, and } x \text{ in } I_n \text{ implies } f_k(x) \geqq b\},$$

$$N_2 = \{n : I_n \text{ has positive endpoints, and } x \text{ in } I_n \text{ implies } f_k(x) \leqq a\},$$

$$N_3 = \{n : I_n \text{ has negative endpoints, and } x \text{ in } I_n \text{ implies } f_k(x) \geqq b\},$$

and

$$N_4 = \{n : I_n \text{ has negative endpoints, and } x \text{ in } I_n \text{ implies } f_k(x) \leqq a\}.$$

Note that

$$N = \bigcup_{i=1}^{4} N_i.$$

Thus, to prove that

$$\lim_{n\to\infty} \mu(I_n)/d(0, I_n) = 0,$$

it need only be shown that

$$N_i\text{-}\lim_{n\to\infty} \mu(I_n)/d(0, I_n) = 0,$$

whenever N_i is an infinite subset of N.

Each of these cases reduces to the special situation already considered. As an example of the reduction process, suppose N_3 is an infinite subset of N. For each n in N_3, if $I_n = [a_n, b_n]$, let $J_n = [-b_n, -a_n]$. Since the endpoints of I_n are negative, those of J_n are positive. In this case, let

$$g(x) = (-1)^k[f(-x) - (f(0) - xf_1(0) + \cdots + (-1)^k(x^k/k!)f_k(0))].$$

From the definition of Peano derivative, it is easy to see that if f has a kth Peano derivative, then the function $h(x) = f(-x)$ does also, and

$$h_j(x) = (-1)^j f_j(-x)$$

for $j = 1, \cdots, k$. Thus,

$$g(0) = g_1(0) = \cdots = g_k(0) = 0,$$

and

$$g_k(x) = (-1)^k[(-1)^k f_k(-x) - (-1)^k f_k(0)]$$
$$= f_k(-x) - f_k(0).$$

If x is in J_n, then $-x$ is in I_n, and, furthermore, $-x$ in I_n implies $f_k(-x) \geqq b$, by the definition of N_3. Therefore, for each n in N_3, x in J_n implies

$$g_k(x) \geqq b - f_k(0) > 0$$

because 0 is in $f_k^{-1}((a,b))$. Since the endpoints of J_n are positive, the conditions of the special case are satisfied by the function g and the sequence $\{J_n\}_{n\in N_3}$. Consequently,

$$N_3\text{-}\lim_{n\to\infty} \mu(J_n)/d(0, J_n) = 0.$$

But, by construction of J_n, $\mu(J_n) = \mu(I_n)$ and $d(0, J_n) = d(0, I_n)$. Thus,

$$N_3\text{-}\lim_{n\to\infty} \mu(I_n)/d(0, I_n) = 0.$$

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